# Distinguishing Knots: The Jones Polynomial and Integer Invariants

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Reading Group

Yulia's Dream

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# What are knots?

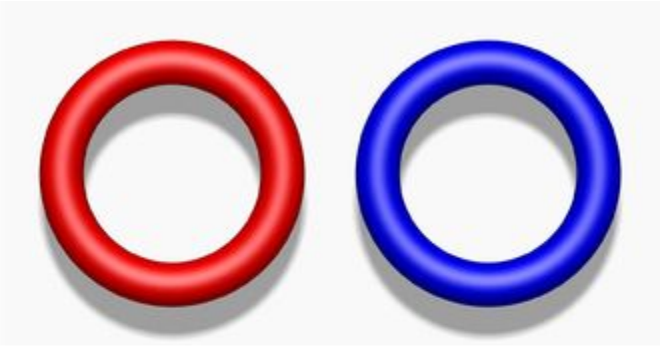
- Knots can be found in everyday life, such as ropes, shoe laces, and DNA molecules.
- A **mathematical knot** is defined as a closed, non-self-intersecting curve that is in three dimensions.

*Creating the Trefoil knot*

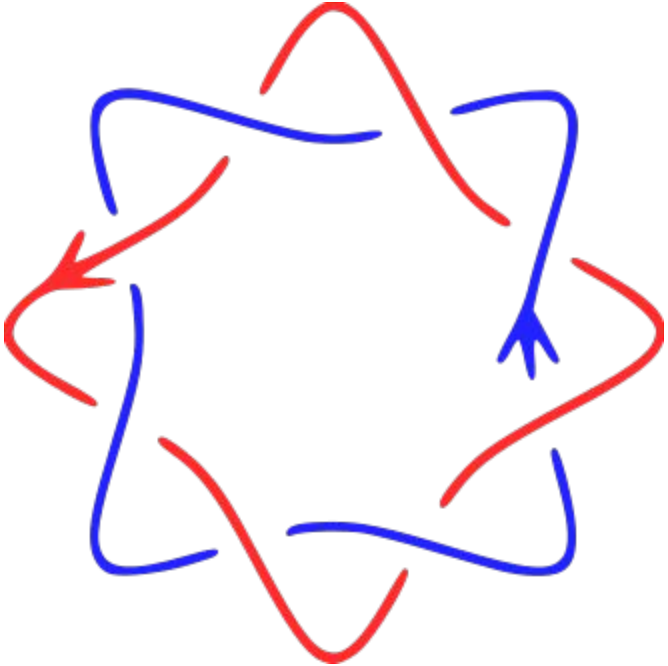
## What is a link?

- A collection of one or more knots that are are considered together is called a **link**.

- A link which is equivalent (under ambient isotopy) to finitely many disjoint circles in the plane called an **unlink**.



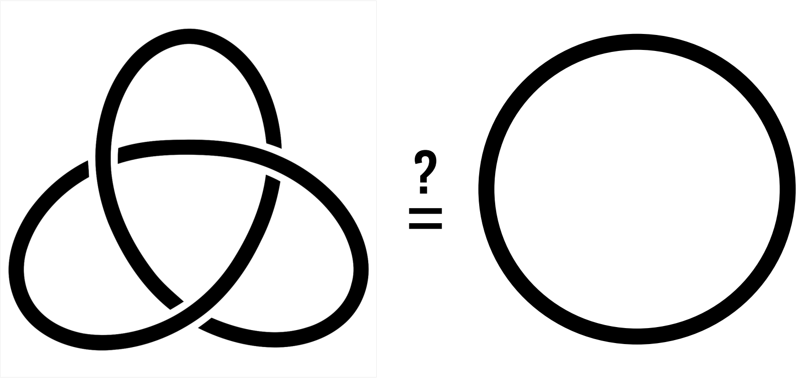
*An unlink of 2 components*



*A 2-component link*

## The Core Problem in Knot Theory

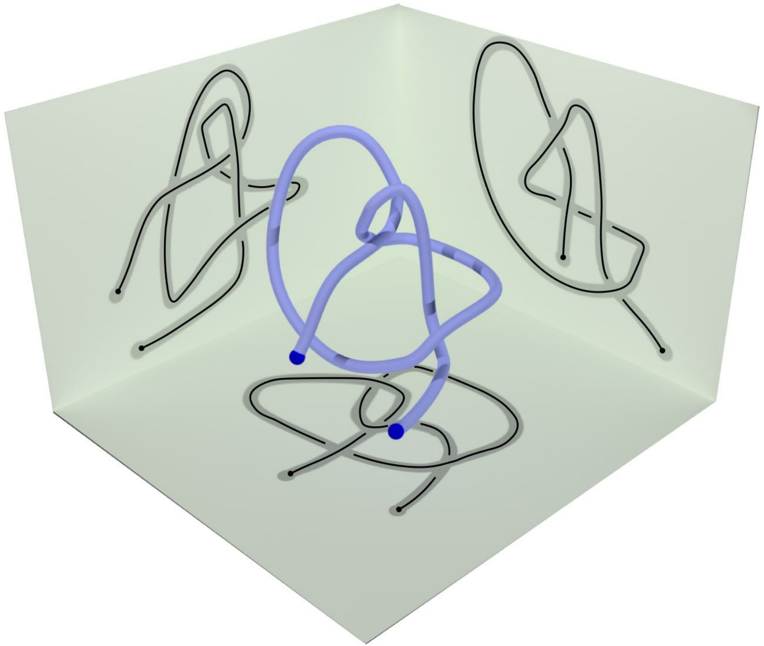
- The core problem in knot theory is **determining whether two links are the same.**
- By definition, two links are the same if one can be transformed into the other without cutting it. Such a transformation is an example of an **ambient isotopy**.



*Is the trefoil knot equivalent to the unknot?*

## Knot Projections

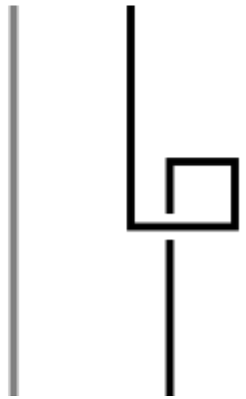
- The pictures of knots before are called **knot projections**. This is a regular projection to some plane in space, in which no three points on the knot project to the same point.
- The places where the knot crosses itself in the 2D picture are called **crossings** of the projection.



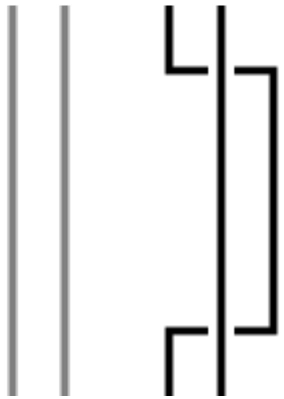
*3 different knot projections*

## The Reidemeister moves

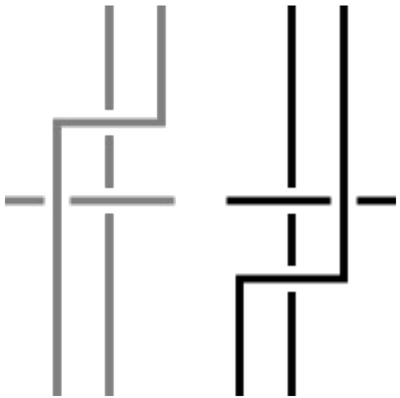
**Theorem (Reidemeister)**
Two links can be smoothly transformed into each other if and only if any projection of one link can be transformed into a projection of the other link through a series of Reidemeister moves.



*Type 1 move*



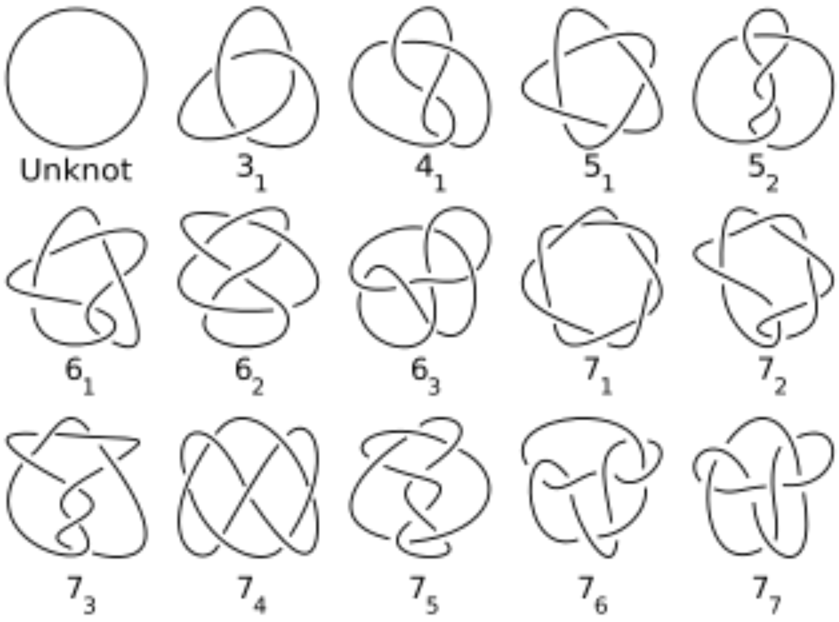
*Type 2 move*



*Type 3 move*

# Knot Invariants

- A knot invariant is a quantity (in a broad sense) defined for each link from some set which is the same for equivalent links (does not change under Reidemeister moves) from this set.



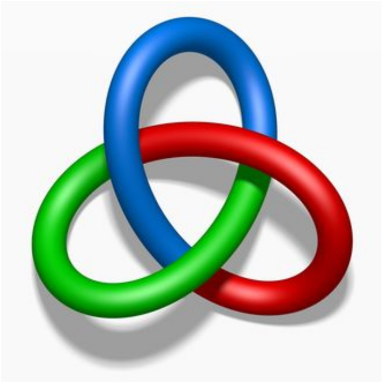

*Some knots are organized by the crossing number invariant*

# Tricolorability

- The **tricolorability** of a knot is the ability of a knot projection to be **colored with three colors** subject to certain rules.

1. At least two colors must be used
2. At each crossing, the three incident strands are either all the same color or all different colors

- As the unknot is not tricolorable, the **trefoil knot is not equal to the unknot**.

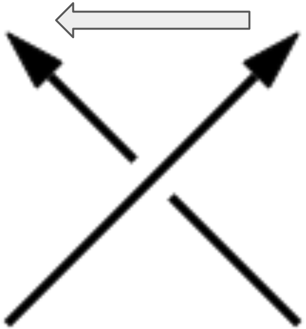


*A tricolored trefoil knot*

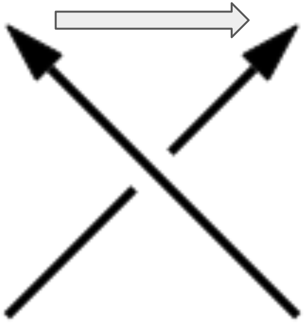
*The unknot*

# Writhe number

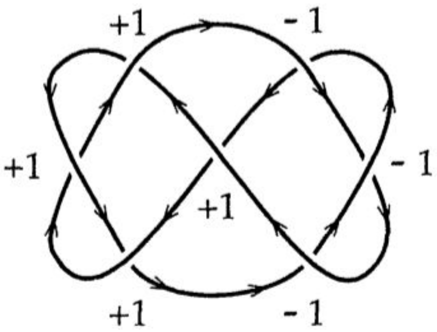
- Define the **writhe** w(L) as the sum of the "crossings" of the link L.
- Only a type 1 Reidemeister move can change w(L) (by +-1).



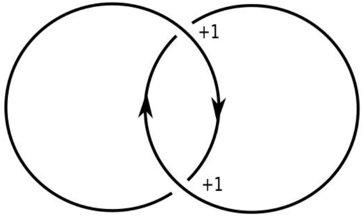
*+1 crossing*



*-1 crossing*




*w(L) = +4-3 = 1*

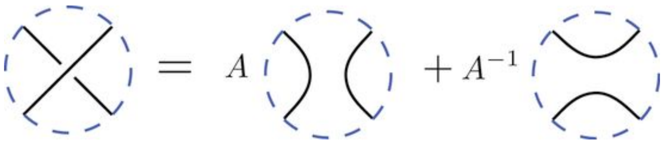



*Example of oriented link*

## Jones polynomial

- Define the **bracket polynomial** <L>, as a polynomial with variable A, characterized by the three rules:

$$\langle \bigcirc \rangle = 1$$ , where $\bigcirc$ is the standard notation of the unknot




$$\langle \bigcirc \cup L \rangle = (-A^2 - A^{-2})\langle L \rangle$$

- Then the **auxiliary polynomial** X(L) is defined as:

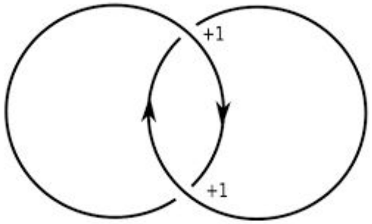
$$X(L) = (-A^3)^{-w(L)}\langle L \rangle$$

- And the **Jones Polynomial** V(L) is obtained from X by replacing each A with $t^{-1/4}$

## Jones polynomial

$$\langle\ \rangle = A \langle\ \rangle + A^{-1} \langle\ \rangle$$
$$= A\,(A\langle\ \rangle + A^{-1} \langle\ \rangle) + A^{-1}\,(A\langle\ \rangle + A^{-1} \langle\ \rangle)$$
$$= A\,(A(-(A^2 + A^{-2})) + A^{-1}\,(1)) + A^{-1}\,(A(1) + A^{-1}\,(-(A^2 + A^{-2})))$$
$$= -A^4 - A^{-4}$$

w(L) = 2




*The writhe number of the Hopf link*

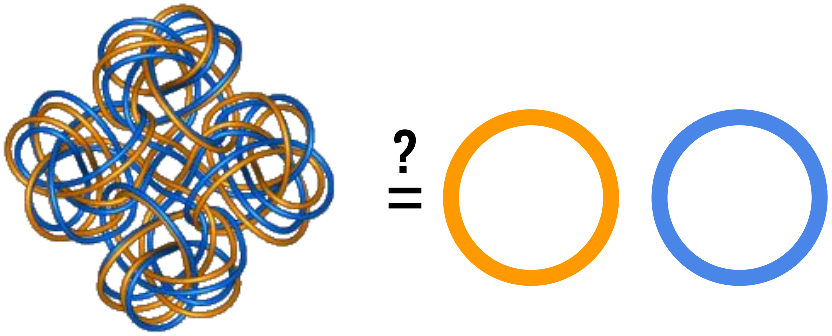
$$X(L) = (-A^3)^{-2}(-A^4 - A^{-4}) = -A^{-2} - A^{-10}$$

$$V(L) = -t^{1/2} - t^{5/2}$$

- An example calculation for the **Hopf link** (replaced with L)

# A Problem with the Jones Polynomial

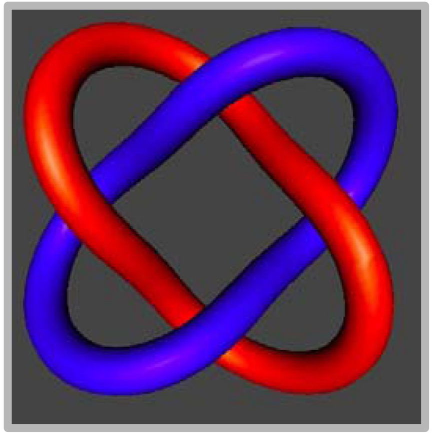
- The Jones Polynomial is **dependent** on the bracket polynomial.
- For every crossing considered, the number of diagrams increase twice. That means the Jones Polynomial becomes **exponentially hard to calculate** ($2^n$).



*Is this link equivalent to an unlink of 2 components?*

# Linking Number

- The linking number is a knot invariant for a 2-component link that measures the **degree of linkage** between two closed curves or components in three-dimensional space.
- Intuitively, it represents the extent to which the two curves are entangled with each other.
- Typically denoted by ***lk(K,J)***, where K and J are knots.

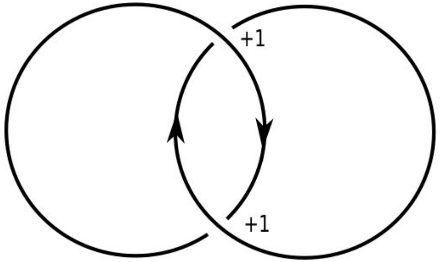


*(2,4)-torus link*

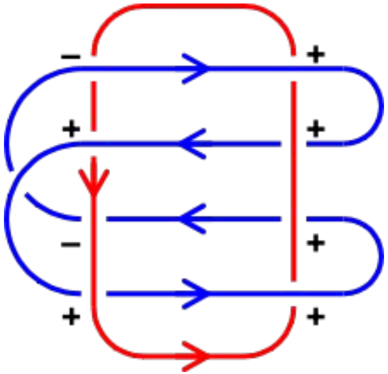
# lk( ) Calculation

- Assign both components with an orientation. Each crossing between both components will be **clockwise** or **counterclockwise**. Give each crossing a value:
  $$c_i = \pm 1$$
- $$lk(K, J) = \frac{1}{2}\sum_{i=1}^{n} c_i$$

*The Hopf link*

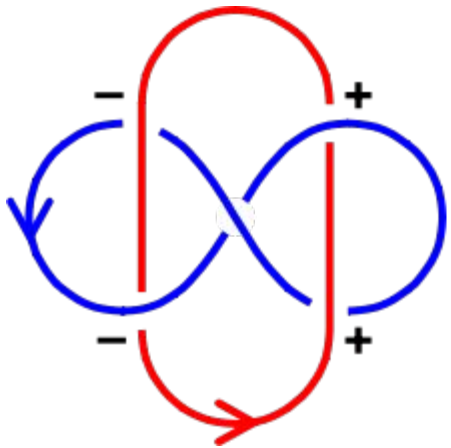



*A link with lk() = +4*



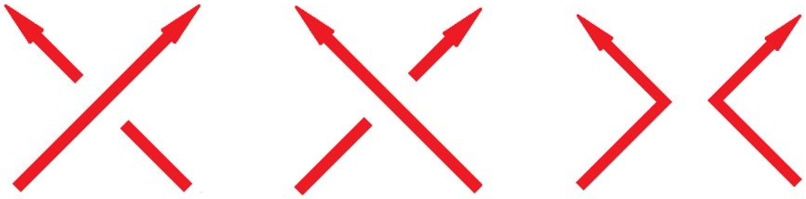
## Limitations of lk( )

- Any link that can be transformed into an unlink has a linking number of 0 but it's not true vice versa.
- An example of this is the **Whitehead link**.
- The linking number can only be computed for **two components** at a time.




*The Whitehead link*

# The Conway Polynomial



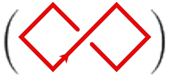
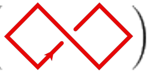
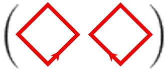
L+ L- L0

- Is denoted by $\nabla(z)$
- Satisfies the following two skein relations:

$$\nabla(O) = 1$$

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$$

- Let's do a simple calculation.

$\nabla($  $) - \nabla($  $) = z\nabla($  $) \Rightarrow \nabla(U_2) = 0$

- Similarly (n > 1): $\nabla(U_n) = 0$

# Properties of its Coefficients (1)

- The Conway Polynomial can be shown in an **expanded form**.

$$\nabla(L) = \sum c_i(L) z^i$$

- After using the skein relation multiple times, we will arrive at a collection of unlinks. The only one which **affects our polynomial is the unknot**.

**Theorem**

For an n-component link L: $c_i(L) = 0$ , for i < n-1.

## Properties of its Coefficients (2)

When L is a:

- 1-component link $c_0(L) = 1$
- 2-component link $c_1(L) = lk(L)$
- 3-component link with linking numbers a, b, and c $c_2(L) = ab + bc + ca$

# Enhanced Linking Number

**Definition**

$$\lambda(L) = c_3(L) - c_1(L)(c_2(K) + c_2(J))$$

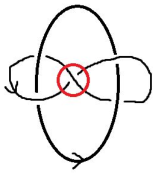
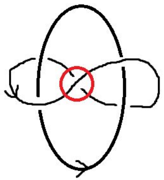
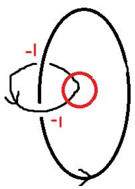
, where K and J are the two knot components of link L.

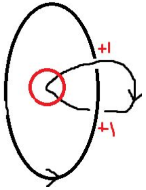
**Theorem (Crossing change formula)**

$\lambda($  $, J) - \lambda($  $, J) = lk($  $, J)\, lk($  $, J)$

## The Whitehead Link

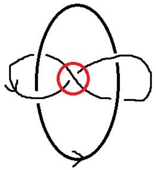
*(This one is easy to compute)*

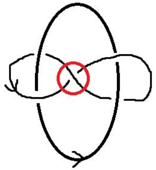
$\lambda($  $) - \lambda($  $) = lk($ 

$)lk($ 

$)$

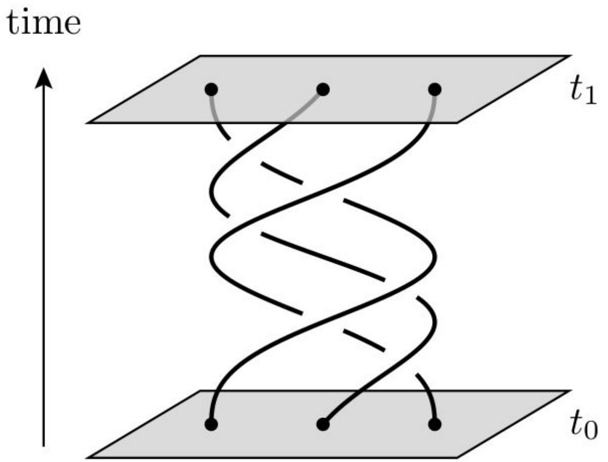
$$\Downarrow$$

$\lambda($  $) - 0 = (-1)(+1)$

$\lambda($  $) = -1$

# Another Approach to Solving the Computation Problem

- A **world-line** is the path a particle takes through time (from $t_0$ to $t_1$)
- Anyons are particles which exist in 2D. Their properties at the end ($t_1$) **depend on the knot/link created**.
- Based on this, the **Jones Polynomial can be calculated** for that knot using the anyons.




*A knot created by the world-lines of anyons*

# Acknowledgements

We would like to thank:

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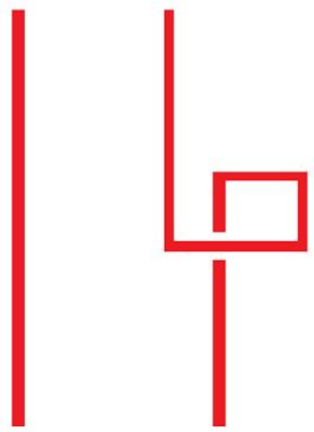
- Dr. Pavel Etingof, Dr. Slava Gerovitch, Dmytro Matvieievskyi and the Yulia's Dream program for providing this valuable opportunity.
- Our families for their constant support.

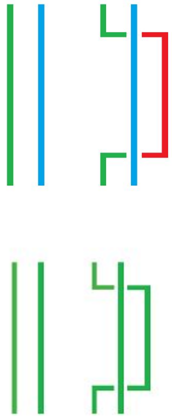
Thank you!

## Proof of invariant

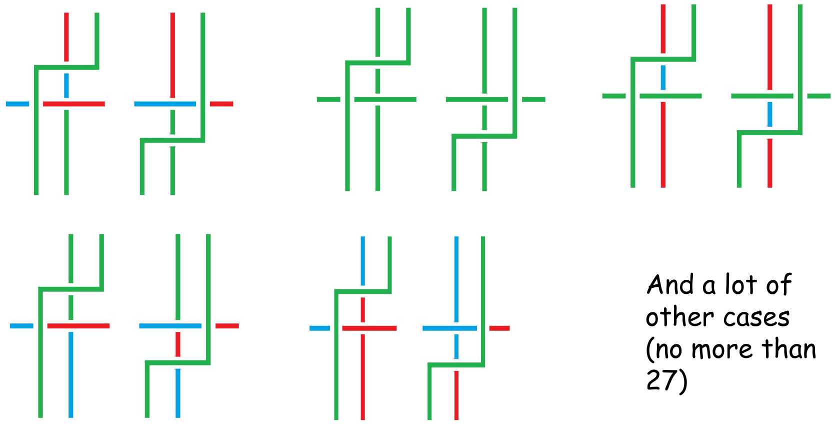
- To prove that tricolorability is an invariant, we must show that it is **not affected by the Reidemeister moves**.



*Type 1 move*



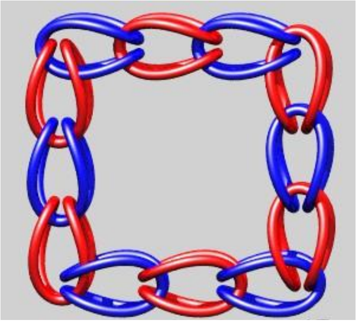
*Type 2 move*



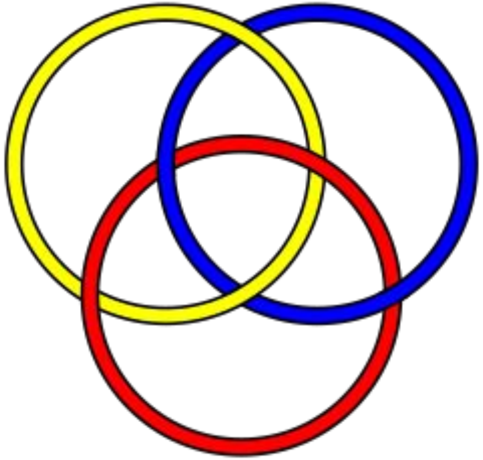

*Type 3 move*

# N-component links

- Links of multiple components can be looked at through a **collection of linking numbers**.
- However it will not work if all of the linking numbers are equal to 0. Such a case are the **Brunnian links**.



*A 12-component Brunnian link*



*The Borromean rings*