52. Austrian Math Olympiads (ÖMO)

Special Math. Olympiad course ”‘Mathematik macht Freu(n)de”’ – Problem sheet for Nov, 28th, 2020

Procedure

This problem sheet was created by Alexander Glazman.
Please send questions and (sketches of) proofs via E-Mail.
We provide hints to selected problems on Nov, 24th, 2020. Alexander Gazman addresses the problems at the virtual course on Nov, 28th, 2020: 13:15–15:00. You can discuss and present your solutions there. Afterwards we provide selected solutions of the problems and give references.
Contact us, if you want to take part in this course. You are always welcome!

Random walks and electric networks

Little Red Cap is wandering around the forest: each minute chooses North, South, East, West uniformly at random and goes in this direction for the next minute. In the middle of the forest is waiting the Big Bad Wolf and at the end is the grandmother. Our main question for today is to show that Little Red Cap will be necessarily eaten by the wolf. However, if she could choose to go also up and down (which suggests that she can fly and dive and the forest turns into the sea), then she has a chance to escape.

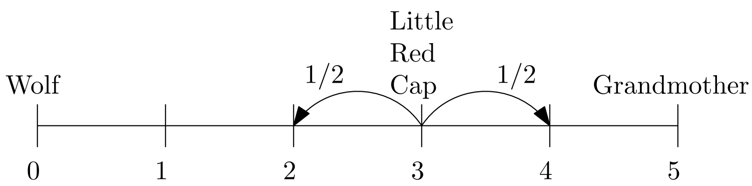


Figure 1: Little Red Cap wandering on a straight path.

We start by a one-dimensional forest of length 5 (Fig. 1): each minute Little Red Cap tosses a fair coin and goes left if it's heads and right if it's tails. At one end of the forest is her grandmother and at the other is Big Bad Wolf. She continues wandering in this way until she either gets safely to her grandmother or gets eaten by the wolf.

Problem 1. Show that, with probability one, Little Red Cap will find her destiny in a finite amount of time.

The process describing wandering Little Red Cap is called a *random walk*. For vertex x, we define by $\mathbb{P}_x$ the random walk starting from x and running until it hits 0 (the wolf) or n (the grandmother's house). Define $h_0 = 0$, $h_n = 1$. Define

$$h_x := \mathbb{P}_x(\text{random walk hits } n \text{ before } 0).$$

Thus defined h_x is called “hitting probability”.

Let A be an event (eg. “Little Red Hat will get to her grandmother's”) which occurs only if event B or event C occurs. Assume B and C are disjoint. Then:

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C).$$

The above is called the *law of total probability*. It is in fact equivalent to considering cases: if B occurs we have probability $\mathbb{P}(A \cap B)$, and if C occurs we have probability $\mathbb{P}(A \cap C)$ (if B and C do not occur, then A neither, by the assumption).

Problem 2. a) Prove that for every $x = 1, 2, \ldots, n-1$, one has that h_x is the mean value of its neighbours:

$$h_x = \tfrac{1}{2}(h_{x-1} + h_{x+1}).$$

Find h_x for every x, depending on n.
b) Let G be a finite graph. For every (ordered!) pair (x, y) of adjacent vertices of G, we have some $P_{xy} \in [0, 1]$, that sum up to 1 at every vertex x:

$$\sum_{y \sim x} P_{xy} = 1.$$

Fix two vertices a and b (the wolf and the grandmother) and define h_x similarly to the above: if the Little Red Cap is at x now, it will next go to one of the neighbours of x; for a vertex y adjacent to x, this probability equals P_{xy}. Show that, for every $x \neq a, b$, we have that $h(x)$ equals mean-value of the neighbours taken with coefficients P_{xy}:

$$h_x = \sum_{y \sim x} P_{xy} h_y. \tag{1}$$

When a function satisfies (1), it is called *harmonic* for $\{P_{xy}\}$. Harmonic functions are crucial in the studies of random walks and satisfy several important properties. The points a and b where the function does not necessarily satisfy the relation (1) should be viewed as the *boundary* (and in case of our first example given by a straight path, this is indeed the boundary).

Problem 3. a) **Maximum principle.** In the notation of Problem 2, let f be any function on the vertices of G that satisfies (1) at every vertex except for a and b. Then the maximum (and minimum) of f is achieved at a or at b.
b) **Uniqueness principle.** Let f and g be two functions on G that satisfy relation (1) and, in addition, $f(a) = g(a)$ and $f(b) = g(b)$. Then functions f and g coincide.

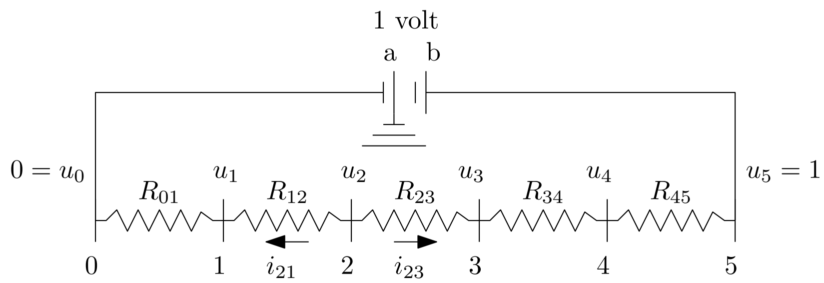


Figure 2: A simple electric circuit of voltage 1 with sources a, b and resistances R_{xy} for $x \sim y$. We ground a and set the total voltage to be 1 volt: $v_a := 0$ and $v_b := 1$. The voltage at point $x \neq a, b$ is denoted by v_x. For neighbours x and y, the current flowing from x to y is denoted by i_{xy} and is called *the current flowing from x to y*.

The following two laws govern the distribution of current.

- **Kirchoff's law.** For every $x \neq 0, n$, the incoming and outgoing current are equal:

$$\sum_{y \sim x} i_{xy} = 0.$$

- **Ohm's law.** For every adjacent vertices x and y, the current is proportional to voltage, with the coefficient given by $1/R_{xy}$ (called conductance):

$$i_{xy} = \frac{v_x - v_y}{R_{xy}}.$$

The conductance is defined as the inverse of the resistance:

$$C_{xy} = \tfrac{1}{R_{xy}}.$$

Problem 4. a) Compute the current at every edge and the voltage every vertex when the network consists of 5 equal resistors put one after the other consecutively.
b) On a general network G with resistances R_{xy}, find the values P_{xy}, for which voltage function is harmonic, that is satisfies (1) for all $x \neq a, b$.
c) Use the properties of harmonic functions to derive that voltages in this electric network (G, R_{xy}) coincide with hitting times of the random walk on (G, P_{xy}).
d) Find the probability interpretation for the current i_{xy}.

The effective resistance of the electric network is defined as the voltage of the network divided by the total current passing through it:

$$R_{eff} := \frac{v_b - v_a}{\sum_{x \sim a} i_{xa}}.$$

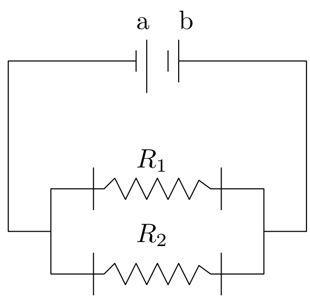
Problem 5. a) Show that

$$R_{eff} \cdot \sum_{x \sim a} \tfrac{1}{R_{ax}} = p_{esc}^{-1},$$

where p_{esc} is the escape probability of the corresponding random walk defined by:

$$p_{esc} := \mathbb{P}(\text{the walk starting at } a \text{ will hit } b \text{ before coming back to } a).$$

b) Compute the effective resistance of the circuits on Fig. 2 (consecutive resistors) and the one below (parallel resistors):



Let us now come back to our original question of Little Red Cap wandering on $\mathbb{Z}^2$ or $\mathbb{Z}^3$. Precisely, we consider a simple random walk (next step is chosen uniformly) on a box $\Lambda_n^d := [-n,n]^d$, for $d = 2,3$. Denote by $p_{esc}(n,d)$ the probability that the walk starting at the origin will get to any vertex at the edge of the box (grandmother) before passing by the origin (wolf). Our goal is then to show:

$$p_{esc}(n,2) \xrightarrow[n\to\infty]{} 0 \quad \text{but} \quad p_{esc}(n,3) \xrightarrow[n\to\infty]{} c > 0.$$

The first random walk is called *recurrent* and the second *transient*.

Problem 6. Describe the electric network corresponding to the above question. What do we need to show about the effective resistance?

In order to construct good lower and upper bounds for the effective resistance on complicated networks, we need to establish several additional properties.In particular, we will show that the current can be characterised as the flow minimising the energy dissipation. We started be the definitions.

The *total energy dissipation* in the circuit is defined by:

$$E := \sum_{x\sim y} i_{xy}^2 R_{xy} = \sum_{x,y} i_{xy}(v_x - v_y),$$

where each edge is counted once (xy or yx), and the second equality follows from Ohm's law.

A flow j from a to b is any assignment of numbers j_{xy} to oriented edges such that,

- $j_{xy} = -j_{xy}$;
- for any $x \neq a, b$, one has $\sum_{y\sim x} j_{xy} = 0$.

Problem 7. a) **Conservation of Energy.** Let w be any function defined at vertices of G and j be a flow from a to b. Then

$$(w_a - w_b)\sum_{x\sim a} j_{ax} = \sum_{x,y}(w_x - w_y)j_{xy}.$$

By taking $j := i$, show that the above translates to:

$$\left(\sum_{x\sim a} i_{ax}\right)^2 R_{eff} = \sum_{x,y} i_{xy}^2 R_{xy},$$

which means that the total energy dissipated is the same when the circuit is replaced with one resistor having resistance R_{eff}.
b) **Thompson's principle.** Assume $v(a) = 0$ and $v(b) = 1$. Consider all flows j with $\sum_{x\sim a} j_{xa} = 1$. Then, among the current i has the minimal total energy dissipation.
c) **Rayleigh's monotonicity law.** If the resistances of a circuit are increased, the effective resistance R_{eff} can only increase.

The above Rayleigh's monotonicity law is a very strong tool in estimating the effective resistance, and hence proving recurrence or transience of a random walk. In fact, the short-cut method would suffice.

Problem 8. a) *Shorting.* Let us merge several vertices. Prove that the effective resistance can only decrease.
b) *Cutting.* Let us erase several edges. Prove that the effective resistance can only increase.

In the remaining exercises all resistors have equal resistance.

Problem 9. a) Prove that, the effective resistance from the origin to the border of a segment $[-n, n]$ on $\mathbb{Z}$ tends to infinity as n increases.
b) The same for the boxes $[-n, n]^2$ on $\mathbb{Z}^2$
c) Derive that the simple random walk in dimensions 1 and 2 is recurrent.

We now start proving transience in $\mathbb{Z}^3$. Here is the first attempt.

Problem 10. a) Let T_2 be a binary tree rooted at the origin. Prove that the effective resistance from the origin to infinity is finite. Derive that the simple random walk on T_2 is transient.
b) Take any dimension d. Prove that there exists no mapping from T_2 to $\mathbb{Z}^d$ that sends different edges to different edges.

The previous exercise shows that a simple random walk on the binary tree is in some sense "more transient" than on $\mathbb{Z}^3$. Hence, we cannot use it to get transience on $\mathbb{Z}^3$ via Rayleigh's law. Still, proving transience using trees could work! Find a suitable tree and solve the last problem:

Problem 11. Show that the effective resistance on $[-n, n]^3$ is bounded above by a constant. Derive that the simple random walk on $\mathbb{Z}^3$ is transient.

Hints to selected problems

Problem 1. Use induction in the length of the forest.

Problem 2. a) Take event $A := \{\text{Little Red Cap ends gets to } n \text{ before } 0\}$. Consider the first step — it is to $x-1$ with probability $1/2$ (event B) and to $x+1$ with probability $1/2$ (event C). Use the law of total probability. The independence of the first step from the rest implies that $\mathbb{P}(A \cap B)$ can be written as a product — $\mathbb{P}(B)$ (=1/2) times the probability that a walk starting from $x-1$ will hit n before 0.

b) Same as in a) but the number of cases (in the law of total probability) equals the degree of the vertex. If the first step is towards y, then the probability is P_{xy} times h_y.

Problem 3. a) Assume the maximum is achieved at $x \neq a, b$. Look at its neighbours and get a contradiction using Eq. (1).

b) Consider $f - g$, show that it is also harmonic, use a).

Problem 4. a) Kirchoff's law gives that the current i_{xy} at every edge is the same. Then Ohm's law then says that the difference of voltages $v_x - v_y$ is also the same. Sum it over all edges and equate with $v_b - v_a = 1$.

b) Plug in Ohm's law into Kirchoff's law. You should get $P_{xy} = \frac{C_{xy}}{\sum_{y \sim x} C_{xy}}$.

c) Use Problem 2b to show that these hitting probabilities form a harmonic function, then use Problem 3b (uniqueness).

d) Start at a and walk randomly until you are in b. The current i_{xy} is related to the probability that you pass the edge from x to y or from y to x, taken with the sign.

Problem 5. a) Compute the escape probability considering possible first steps and using the law of total probability. In each case, you will have a product of two terms: $P_{ax} \cdot h_x$. Then expand each of them using the interpretation via electric currents in Problem 4.
b) Use Kichoff's law and Ohm's law to get a system of equations.

Problem 6. Graph: a box of size n with the whole boundary merged in one vertex (we keep multiple edges). Resistors: all equal. We're interested in estimating the escape probability.

Problem 7. a) Expansion using the definition of a flow.

b) Let j_{xy} be any flow. Define $d_{xy} := j_{xy} - i_{xy}$. Then d is a flow of with $\sum_{x \sim a} d_{ax} = 0$. Expand the total energy dissipation for j via i and d. Use Ohm's law for i and energy conservation for d.

c) Take the current i for the original circuit and current j for the circuit with modified resistors. For both circuits the expression of the effective resistance as the total energy dissipated. Use Thompson's principle to compare the two (view j as a flow in the original circuit).

Problem 8. a) Shorting means setting resistance to 0.

b) Cutting means setting resistance to infinity.

Problem 9. a) The effective resistance here is the sum of resistances over all edges.

b) It is enough to prove this after the shorting operation: for all k, merge all vertices on the boundary of a box $[-k, k]^2$ into one vertex.

c) Use Problems 5a and 6.

Problem 10. a) Compute the effective resistance by induction, use Problem 5b.

b) Consider the number of edges at distance k from the origin.

Problem 11. Consider the tree that has three branches starting from the origin. Each branch splits in three new branches when the distance from the origin equals $2^n - 1$, for some n.

Proposed solution to selected problems

Solutions to the exercises and further details may be found at [1].

References

[1] Peter G Doyle and J Laurie Snell. *Random walks and electric networks*, volume 22. American Mathematical Soc., 1984.