53. Österreichische Mathematik-Olympiade

Kurs für Internationale „Mathematik macht Freu(n)de“ – Aufgabenblatt für den 23. Oktober 2021

Ablauf

Dieses Aufgabenblatt wurde von Alexander Glazman zusammengestellt.

Wir freuen uns auf deine Fragen und Lösungsvorschläge per E-Mail.

Am 20. Oktober 2021 wird das Blatt mit Tipps zur Lösung ausgewählter Aufgaben ergänzt. Alexander Glazman trägt zum Thema im virtuellen Olympiade-Kurs am 23. Oktober 2021 von 10:00–11:45 Uhr vor. Kurz darauf ergänzen wir das Blatt um ausgewählte Lösungsvorschläge und Angaben zu den Quellen der Aufgaben.

Schreibe uns, wenn du bei den virtuellen Kursen dabei sein möchtest. Du bist jederzeit willkommen!

Mass point geometry

Aufgaben

Consider a system of points on the plane together with masses assigned to them. In other words, we have a collection (X_1, m_1), (X_2, m_2), …, (X_n, m_n), where X_i are points on the plane and m_i is a strictly positive real number (a mass). We say that the point O is the center of mass of this system if

$$m_1\overrightarrow{OX_1} + \cdots + m_n\overrightarrow{OX_n} = 0.$$

Aufgabe 1. a) Let O be a center of mass and let X be any point on the plane. Show that

$$\overrightarrow{XO} = \frac{1}{m_1 + \cdots + m_n} \cdot (\overrightarrow{XX_1} + \cdots + \overrightarrow{XX_n}).$$

b) Show that the center of mass always exists and is unique.

c) Show that the center of mass of two points (X_1, m_1) and (X_2, m_2) is the point O on the segment X_1X_2 such that

$$\frac{|OX_1|}{|OX_2|} = \frac{m_2}{m_1}.$$

Note the above reasoning works also if some masses are negative — as long as the sum of all masses is non-zero.
A crucial property of the center of mass is that it does not change when a group of points is replaced by their center of masses caring their total mass!

Aufgabe 2. Consider a system of points (X_1, m_1), …, (X_n, m_n) and take any $k < n$. Let O_k be the center of mass of (X_1, m_1), …, (X_k, m_k) and take $m = m_1 + \cdots + m_k$. Consider a new system points: (O_k, m), (X_{k+1}, m_{k+1}), …, (X_n, m_n). Show that the center of mass of both systems of points is the same.

Aufgabe 3. a) Show that the medians of a triangle ABC intersect at the same point. Moreover, this point divides each of the medians as $2 : 1$. This point is usually referred to as the *center of mass of the triangle ABC*.

b) Let $ABCD$ be a quandrilateral. Let M_{AB}, M_{BC}, M_{CD}, M_{DA} be the middles of the corresponding sides of $ABCD$, and let M_{AC} and M_{BD} be the midpoints of the diagonals. Show that midpoints of the segments $M_{AB}M_{CD}$, $M_{BC}M_{DA}$, and $M_{AC}M_{BD}$ coincide.

c) Let $ABCDEF$ be any hexagon. Let M_{AB}, M_{BC}, M_{CD}, M_{DE}, M_{EF}, and M_{FA} be the midpoints of the corresponding sides of $ABCDEF$. Show that all six medians in the triangles $M_{AB}M_{CD}M_{EF}$ and $M_{BC}M_{DE}M_{FA}$ intersect at the same point.

Aufgabe 4. Three flies of equal weight are crawling along the sides of a triangle ABC in such a way that their center of mass stays the same. Assume that one of the flies eventually visits each point on the sides of the triangle. Show that the center of mass of the flies coincides with the center of mass of ABC.

Aufgabe 5. Let ABC be a triangle whose sides BC, CA, AB have lengths a, b, c, respectively. Put masses a, b, c at midpoints of the corresponding sides. Where is the center of mass of this system?

Note the system in the previous exercise is equivalent (with proper definitions) to considering the center of mass of a triangle whose sides are made out of some material and have the same thickness.

Aufgabe 6. Fix n points on the unit circle and assign to them equal masses. For any $n-2$ of these points, consider the line that passes through their center of mass and is perpendicular to the segment linking the other two points. Show that all these lines pass through intersect at the same point.

Let (X_1, m_1), $\dots$, (X_n, m_n) be a system of point masses. For any point P, we define the *moment of inertia* by

$$I_P^2 = m_1|PX_1|^2 + \cdots + m_n|PX_n|^2.$$

Aufgabe 7. Let O be the center of mass of the system and let P be any point. Show that

$$I_P = I_O + (m_1 + \cdots + m_n)|OP|^2.$$

Aufgabe 8. a) Consider points X_1, $\dots$, X_n and put mass 1 at every of them. Denote their center of mass by O. Show that

$$I_O = \frac{1}{n} \cdot \sum_{i<j} |X_iX_j|^2.$$

b) Consider n points inside a unit circle. Show that the sum of the squares of distances between these points is not greater than n. When is it equal to n?

Fix a triangle ABC in the plane. Place at the vertices some masses m_a, m_b, m_c. If M is the center of mass of this system, then we say that $(m_a : m_b : m_c)$ are **baricentric coordinates** of M with respect to the triangle ABC. These coordinates can be very useful in solving problems in geometry in olympiads.

Aufgabe 9. a) Show that every point has some baricentric coordinates.
b) Under the restriction that $m_a + m_b + m_c = 1$, the baricentric coordinates of a given point M are defined uniquely.

c) Show that if a point X lies inside of ABC, then its baricentric coordinates are given by areas of the three triangles in which it splits ABC: $(S_{BCX} : S_{CAX} : S_{ABX})$. What to do if X lies outside of ABC?

Below are two examples how barycentric coordinates can be used.

Aufgabe 10. Let ABC be a triangle such that $|AB| \neq |AC|$. Pick any point K on the bisector of the (inner) angle A. Denote by L and M the points in which the lines BK and CK intersect AC and AB, respectively. Show that the line LM passes through some fixed point that does not depend on the choice of K.

Aufgabe 11. Consider a circle ω centered at O and let A, B, C be three points on the circle. Point P lies on line BC such that line PA is tangent to ω. The bisector of $\angle APB$ intersects segments AB and AC at D and E, respectively. Let Q be the intersection point of segments BE and CD. Assume that O, P, Q belong to the same line. Find $\angle BAC$.