53. Österreichische Mathematik-Olympiade

Kurs für Internationale „Mathematik macht Freu(n)de“ – Aufgabenblatt für den 23. Oktober 2021

Ablauf

Dieses Aufgabenblatt wurde von Alexander Glazman zusammengestellt.

Wir freuen uns auf deine Fragen und Lösungsvorschläge [per E-Mail].


[Schreibe uns] wenn du bei den virtuellen Kursen dabei sein möchtest. Du bist jederzeit willkommen!

Mass point geometry

Aufgaben

Consider a system of points on the plane together with masses assigned to them. In other words, we have a collection \((X_1, m_1), (X_2, m_2), \ldots, (X_n, m_n)\), where \(X_i\) are points on the plane and \(m_i\) is a strictly positive real number (a mass). We say that the point \(O\) is the center of mass of this system if

\[
m_1\overrightarrow{OX}_1 + \cdots + m_n\overrightarrow{OX}_n = 0.
\]

**Aufgabe 1.**

a) Let \(O\) be a center of mass and let \(X\) be any point on the plane. Show that

\[
\overrightarrow{XO} = \frac{1}{m_1 + \cdots + m_n} \cdot (\overrightarrow{XX}_1 + \cdots + \overrightarrow{XX}_n).
\]

b) Show that the center of mass always exists and is unique.

c) Show that the center of mass of two points \((X_1, m_1)\) and \((X_2, m_2)\) is the point \(O\) on the segment \(X_1X_2\) such that

\[
\frac{|OX_1|}{|OX_2|} = \frac{m_2}{m_1}.
\]

Note the above reasoning works also if some masses are negative — as long as the sum of all masses is non-zero.

A crucial property of the center of mass is that it does not change when a group of points is replaced by their center of masses caring their total mass!

**Aufgabe 2.** Consider a system of points \((X_1, m_1), \ldots, (X_n, m_n)\) and take any \(k < n\). Let \(O_k\) be the center of mass of \((X_1, m_1), \ldots, (X_k, m_k)\) and take \(m = m_1 + \cdots + m_k\). Consider a new system points: \((O_k, m), (X_{k+1}, m_{k+1}), \ldots, (X_n, m_n)\). Show that the center of mass of both systems of points is the same.
Aufgabe 3. a) Show that the medians of a triangle $ABC$ intersect at the same point. Moreover, this point divides each of the medians as $2 : 1$. This point is usually referred to as the **center of mass** of the triangle $ABC$.

b) Let $ABCD$ be a quadrilateral. Let $M_{AB}, M_{BC}, M_{CD}, M_{DA}$ be the middles of the corresponding sides of $ABCD$, and let $M_{AC}$ and $M_{BD}$ be the midpoints of the diagonals. Show that midpoints of the segments $M_{AB}M_{CD}, M_{BC}M_{DA}$, and $M_{AC}M_{BD}$ coincide.

c) Let $ABCDEF$ be any hexagon. Let $M_{AB}, M_{BC}, M_{CD}, M_{DE}, M_{EF},$ and $M_{FA}$ be the midpoints of the corresponding sides of $ABCDEF$. Show that all six medians in the triangles $M_{AB}M_{CD}M_{EF}$ and $M_{BC}M_{DE}M_{FA}$ intersect at the same point.

Aufgabe 4. Three flies of equal weight are crawling along the sides of a triangle $ABC$ in such a way that their center of mass stays the same. Assume that one of the flies eventually visits each point on the sides of the triangle. Show that the center of mass of the flies coincides with the center of mass of $ABC$.

Aufgabe 5. Let $ABC$ be a triangle whose sides $BC, CA, AB$ have lengths $a, b, c$, respectively. Put masses $a, b, c$ at midpoints of the corresponding sides. Where is the center of mass of this system? Note the system in the previous exercise is equivalent (with proper definitions) to considering the center of mass of a triangle whose sides are made out of some material and have the same thickness.

Aufgabe 6. Fix $n$ points on the unit circle and assign to them equal masses. For any $n-2$ of these points, consider the line that passes through their center of mass and is perpendicular to the segment linking the other two points. Show that all these lines pass through intersect at the same point.

Let $(X_1, m_1), \ldots, (X_n, m_n)$ be a system of point masses. For any point $P$, we define the moment of inertia by

$$I_P = m_1|PX_1|^2 + \cdots + m_n|PX_n|^2.$$ 

Aufgabe 7. Let $O$ be the center of mass of the system and let $P$ be any point. Show that

$$I_P = I_O + (m_1 + \cdots + m_n)|OP|^2.$$ 

Aufgabe 8. a) Consider points $X_1, \ldots, X_n$ and put mass $1$ at every of them. Denote their center of mass by $O$. Show that

$$I_O = \frac{1}{n} \sum_{i<j} |X_iX_j|^2.$$ 

b) Consider $n$ points inside a unit circle. Show that the sum of the squares of distances between these points is not greater than $n$. When is it equal to $n$?

Fix a triangle $ABC$ in the plane. Place at the vertices some masses $m_a, m_b, m_c$. If $M$ is the center of mass of this system, then we say that $(m_a : m_b : m_c)$ are **baricentric coordinates** of $M$ with respect to the triangle $ABC$. These coordinates can be very useful in solving problems in geometry in olympiads.

Aufgabe 9. a) Show that every point has some baricentric coordinates.

b) Under the restriction that $m_a + m_b + m_c = 1$, the baricentric coordinates of a given point $M$ are defined uniquely.
c) Show that if a point $X$ lies inside of $ABC$, then its barycentric coordinates are given by areas of the three triangles in which it splits $ABC$: $(S_{BCX} : S_{CAX} : S_{ABX})$. What to do if $X$ lies outside of $ABC$?

Below are two examples how barycentric coordinates can be used.

**Aufgabe 10.** Let $ABC$ be a triangle such that $|AB| \neq |AC|$. Pick any point $K$ on the bisector of the (inner) angle $A$. Denote by $L$ and $M$ the points in which the lines $BK$ and $CK$ intersect $AC$ and $AB$, respectively. Show that the line $LM$ passes through some fixed point that does not depend on the choice of $K$.

**Aufgabe 11.** Consider a circle $\omega$ centered at $O$ and let $A, B, C$ be three points on the circle. Point $P$ lies on line $BC$ such that line $PA$ is tangent to $\omega$. The bisector of $\angle APB$ intersects segments $AB$ and $AC$ at $D$ and $E$, respectively. Let $Q$ be the intersection point of segments $BE$ and $CD$. Assume that $O, P, Q$ belong to the same line. Find $\angle BAC$. 