Level 5 Module 2
exemplar with comments

2014-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2014

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)

Question-Answer Book

8.30 am – 11.00 am (2½ hours)
This paper must be answered in English

INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11, 13 and 15.

2. Answer **ALL** questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

3. Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string **INSIDE** this Book.

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers must be exact.

6. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.

7. The diagrams in this paper are not necessarily drawn to scale.

8. No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

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FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Answers written in the margins will not be marked.

Section A (50 marks)

1. In the expansion of $(1-4x)^2(1+x)^n$, the coefficient of x is 1.

   (a) Find the value of n.

   (b) Find the coefficient of x^2.

(4 marks)

(a) $(1-4x)^2(1+x)^n = (1-8x+16x^2)(1+nx+C_2^n x^2+\dots)$

$\therefore$ coefficient of $x = n-8$

$1 = n-8$

$n = 9$

(b) coefficient of $x^2 = 16 + (-8)n + C_2^n$

$= -20$

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2. Consider the curve $C: y = x^3 - 3x$.

(a) Find $\frac{dy}{dx}$ from first principles.

(b) Find the range of x where C is decreasing.

(5 marks)

(a) let $f(x)$ be x^3-3x

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^3 - 3(x+\Delta x)] - (x^3-3x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - \cancel{3x} - 3\Delta x \cancel{-x^3} \cancel{+3x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 - 3$$

$$= 3x^2 - 3$$

(b) for $\frac{dy}{dx} = 0$, $x = 1$ or -1

	$x<-1$	$x=-1$	$-1<x<1$	$x=1$	$x>1$
$\frac{dy}{dx}$	+	0	−	0	+

$\therefore$ ~~from~~ C is decreasing for $-1<x<1$

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3. Find the equation of tangent to the curve $x\ln y + y = 2$ at the point where the curve cuts the y-axis.

(5 marks)

$$x\ln y + y = 2$$

$$\ln y + x\left(\frac{1}{y}\right)\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(\frac{x}{y}+1\right) = -\ln y$$

$$\frac{dy}{dx} = \frac{-\ln y}{\frac{x}{y}+1}$$

$$\frac{dy}{dx} = -\frac{y\ln y}{x+y}$$

For $x=0$, $y=2$ $\therefore$ y-intercept is 2

slope of tangent : $\left.\frac{dy}{dx}\right|_{(0,2)} = -\frac{2\ln 2}{0+2}$

$= -\ln 2$

eqn of tangent: $y-2 = (-\ln 2)(x)$

$y = -\ln 2\, x + 2$

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4. Let $x = 2y + \sin y$. Find $\frac{d^2y}{dx^2}$ in terms of y.

(3 marks)

$$x = 2y + \sin y$$

$$1 = 2\frac{dy}{dx} + \cos y \left(\frac{dy}{dx}\right)$$

$$1 = \frac{dy}{dx}(2 + \cos y)$$

$$\frac{dy}{dx} = (2 + \cos y)^{-1}$$

$$\frac{d^2y}{dx^2} = -(2 + \cos y)^{-2}(-\sin y)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\sin y}{(2 + \cos y)^2} \cdot \left(\frac{1}{2 + \cos y}\right)$$

$$\frac{d^2y}{dx^2} = \frac{\sin y}{(2 + \cos y)^3}$$

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5. (a) Find $\int \frac{dx}{\sqrt{9-x}}$, where $x < 9$.

(b) Using integration by substitution, find $\int \frac{dx}{\sqrt{9-x^2}}$, where $-3 < x < 3$.

(6 marks)

(a) $\int \frac{1}{\sqrt{9-x}}\,dx = -2\int \frac{\sqrt{9-x}}{\sqrt{9-x}}\,d\sqrt{9-x}$

$= -2\int d\sqrt{9-x}$

$= -2\sqrt{9-x} + C_1$

(b) Sub $x = 3\sin\theta$ $\quad dx = 3\cos\theta\,d\theta$

$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\,d\theta$

$= \int \frac{3\cos\theta}{\sqrt{9}\sqrt{\cos^2\theta}}\,d\theta$

$= \int \frac{3\cos\theta}{3\cos\theta}\,d\theta$

$= \int d\theta$

$= \theta$

$= \sin^{-1}\frac{x}{3} + C_2$

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6. (a) Find $\int xe^{-x}\mathrm{d}x$.

(b)

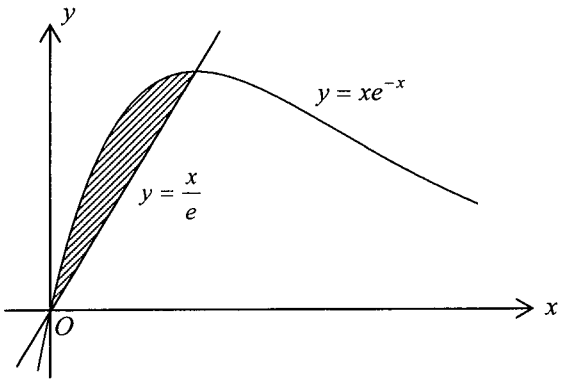

Figure 1

Figure 1 shows the shaded region bounded by the curve $y = xe^{-x}$ and the straight line $y = \frac{x}{e}$. Find the area of the shaded region.

(6 marks)

(a) $\int xe^{-x}dx = -\int x \quad de^{-x}$

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - \int e^{-x}\, d(-x)$

$= -\frac{x}{e^x} - e^{-x} + C$

(b) $\frac{x}{e} = xe^{-x}$

$x = xe^{-x+1}$

$0 = xe^{-x+1} - x$

$x(e^{1-x} - 1) = 0$

$x = 0$ or $e^{1-x} = 1$

$(1-x)\ln e = \ln 1$

$1 - x = \ln 1$

$x = 1 - \ln 1$

$x = 1$

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$\therefore$ Area $= \int_0^1 \left(xe^{-x} - \frac{x}{e}\right) dx$

$= \int_0^1 xe^{-x}\, dx - \int_0^1 \frac{x}{e}\, dx$

$= \left[-\frac{x}{e^x} - e^{-x}\right]_0^1 - \frac{1}{e}\int_0^1 x\, dx$

$= \left(-\frac{1}{e} - \frac{1}{e}\right) - \left(0 - \frac{1}{e^0}\right) - \frac{1}{e}\left[\frac{x^2}{2}\right]_0^1$

$= -\frac{2}{e} + 1 - \frac{1}{e}\left(\frac{1}{2} - 0\right)$

$= -\frac{4}{2e} + 1 - \frac{1}{2e}$

$= 1 - \frac{5}{2e}$

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7. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) Prove, by mathematical induction, that for all positive integers n, $A^{n+1} = 2^n A$.

(b) Using the result of (a), Willy proceeds in the following way:

$$A^2 = 2A$$
$$A^2 A^{-1} = 2AA^{-1}$$
$$A = 2I$$

Explain why Willy arrives at a wrong conclusion.

(7 marks)

(a) For $n=1$,

$$L.H.S. = A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$R.H.S. = 2A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\therefore$ L.H.S. = R.H.S. the statement is true for $n=1$

Assume $A^{k+1} = 2^k A$ where k is a +ve integer.

$$A^{(k+1)+1} = A^{k+1} \cdot A$$
$$= 2^k A \cdot A$$
$$= 2^k \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$
$$= 2^{k+1} \cdot 2^{-1} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$
$$= 2^{k+1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$
$$= 2^{k+1} A$$

$\therefore$ the statement is also true for $n = k+1$.

By the principle of MI, the statement is true for all +ve integers n.

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(b) let $A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0$

$\therefore$ A is ~~no~~ sigular, A^{-1} does not exists.

$\therefore$ the 2nd step of Willy's deduction is wrong and therefore he arrives at a wrong conclusion.

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8. Let $\overrightarrow{OP} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OR} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OQ}$.
Hence find the volume of tetrahedron $OPQR$.

(b) Find the acute angle between the plane OPQ and the line OR , correct to the nearest $0.1°$.

(8 marks)

(a) $\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 6\vec{i} + 4\vec{j} - \vec{k}$

$\overrightarrow{OR} \cdot (\overrightarrow{OP} \times \overrightarrow{OQ}) = (2\vec{i} - 3\vec{j} + 6\vec{k})(6\vec{i} + 4\vec{j} - \vec{k})$

$= 12 - 12 - 6$

$= -6$

$\therefore$ volume req. $= \frac{1}{6}|-6| = 1$

(b) $\overrightarrow{OR} \cdot (\overrightarrow{OP} \times \overrightarrow{OQ}) = -6$

$\therefore |\overrightarrow{OR}||\overrightarrow{OP} \times \overrightarrow{OQ}| \cos\theta = -6$ where θ is the angle between $\overrightarrow{OR}$ and $\overrightarrow{OP} \times \overrightarrow{OR}$

$|\overrightarrow{OR}| = \sqrt{2^2 + (-3)^2 + 6^2} = 7$

$|\overrightarrow{OP} \times \overrightarrow{OQ}| = \sqrt{6^2 + 4^2 + (-1)^2} = \sqrt{53}$

$\therefore 7\sqrt{53} \cos\theta = -6$

$\theta = 96.8°$ (corr to nearest 0.1°)

$\therefore$ the acute angle between OPQ and OR

$= 180° - \theta$

$= 83.2°$ (corr to the nearest 0.1°)

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9. (a) Solve the system of linear equations $\begin{cases} x + y + z = 100 \\ x + 6y + 10z = 200 \end{cases}$.

(b) In a store, the prices of each of small, medium and large marbles are \$0.5, \$3 and \$5 respectively. Aubrey plans to spend all \$100 for exactly 100 marbles, which include m small marbles, n medium marbles and k large marbles.
Aubrey claims that there is only one set of combination of m, n and k. Do you agree? Explain your answer.

(6 marks)

(a) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 1 & 6 & 10 & 200 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 5 & 9 & 100 \end{array}\right)$

let z be t,

$5y + 9t = 100$

$y = \frac{100-9t}{5}$

$x + \frac{100-9t}{5} + t = 100$

$x = 100 - t - \frac{100-9t}{5}$

$x = \frac{500 - 5t - 100 + 9t}{5}$

$x = \frac{400+4t}{5}$

$\therefore (x, y, z) = \left(\frac{400+4t}{5}, \frac{100-9t}{5}, t\right)$ for any real no. t.

(b) $\begin{cases} 0.5m + 3n + 5k = 100 \\ m + n + k = 100 \end{cases}$

$\therefore \left(\begin{array}{ccc|c} m & +n & +k & 100 \\ m & +6n & +10k & 200 \end{array}\right)$

by (a), there are infinitely many solutions for the values of m, n and k.

$\frac{400+4k}{5} = 80 + \frac{4}{5}k$

$\frac{100-9k}{5} = 20 - \frac{9}{5}k$

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$\therefore$ for $k = 5M$ where M is a +ve integer

value of m and n would be integers.

for $M > 3$, $n < 0$ which is impossible

$\therefore$ the only 2 sets of solutions are

$(m, n, k) = (84, 11, 5)$ or $(80, 2, 10)$

$\therefore$ Aubrey is incorrect.

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Section B (50 marks)

10.

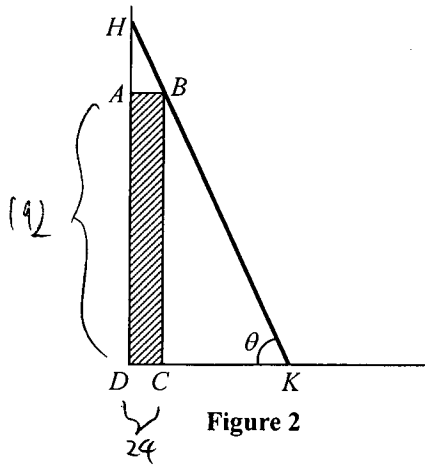

Figure 2

Thomas has a bookcase of dimensions 100 cm × 24 cm × 192 cm at the corner in his room. He wants to hang a decoration on the wall above the bookcase. Therefore, he finds a ladder to climb up. Initially, the ladder touches the wall, the edge of the top of the bookcase and the floor at the same time. Let rectangle $ABCD$ be the side-view of the bookcase and HK be the side-view of the ladder, so that $AB = 24$ cm and $BC = 192$ cm (see Figure 2). Let $\angle HKD = \theta$.

(a) Find the length of HK in terms of θ .

(1 mark)

(b) Prove that the shortest length of the ladder is $120\sqrt{5}$ cm .

(5 marks)

(c)

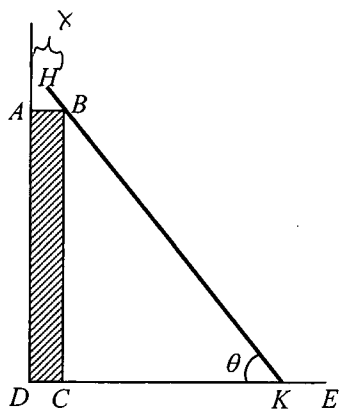

Figure 3

Suppose the length of the ladder is 270 cm . Suddenly, the ladder slides down so that the end of the ladder, K, moves towards E (see Figure 3). The ladder touches the edge of the top of the bookcase and the floor at the same time. Let x cm and y cm be the horizontal distances from H and K respectively to the wall.

(i) When $CK = 160$ cm , the rate of change of θ is -0.1 rad s^{-1} . Find the rate of change of x at this moment, correct to 4 significant figures.

(ii) Thomas claims that K is moving towards E at a speed faster than the horizontal speed H is leaving the wall. Do you agree? Explain your answer.

(6 marks)

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(a) $\tan\theta = \frac{192}{CK}$

$CK = \frac{192}{\tan\theta}$

$\therefore DK = 24 + \frac{192}{\tan\theta}$

$\cos\theta = \frac{DK}{HK}$

$HK = \frac{24 + \frac{192}{\tan\theta}}{\cos\theta}$

$HK = 24\sec\theta + 192\csc\theta$

(b) $\frac{dHK}{d\theta} = 24\tan\theta\sec\theta - 192\cot\theta\csc\theta$

when $\frac{dHK}{d\theta} = 0$, $24\tan\theta\sec\theta = 192\cot\theta\csc\theta$

$\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right) = (8)\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\sin\theta}\right)$

$\sin^3\theta = 8\cos^3\theta$

$\tan^3\theta = 8$

$\tan\theta = 2$

$\theta = 1.11$

	$0<\theta<1.11$	$\theta=1.11$	$1.11<\theta<\frac{\pi}{2}$
$\frac{dHK}{d\theta}$	$-$	0	$+$

$\therefore$ min value of $HK = 24\sec 1.11 + 192\csc 1.11 = 120\sqrt{5}$

$\therefore$ shortest length is $120\sqrt{5}$ cm.

(c) (i) when $CK = 160$, $\theta = 0.876$ rad

$\sin\theta = \frac{192}{BK}$

$BK = 192\csc\theta$

$AB = 270 - BK$

$= 270 - 192\csc\theta$

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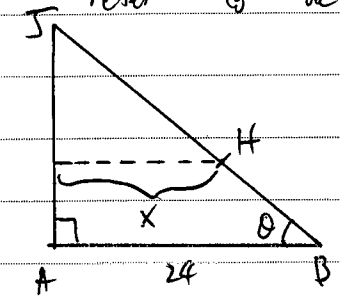
(Q11) refer to the following figure



$\cancel{\tan\theta = \frac{AJ}{24}}$

$\cancel{AJ = 24\tan\theta}$

$\cancel{\cos\theta = \frac{BJ}{24}}$ $\quad \cos\theta = \frac{24}{BJ}$

$BJ = 24\sec\theta$

$\therefore JH = 24\sec\theta - 270 + 192\csc\theta$

$\frac{x}{24} = \frac{JH}{JB}$

$x = \frac{24JH}{JB}$

$\frac{dx}{dt} = 24\frac{d}{dt}\frac{24\sec\theta - 270 + 192\csc\theta}{24\sec\theta}$

$\frac{dx}{dt} = \frac{d\theta}{dt}24 \times \frac{(24\sec\theta)(24\tan\theta\sec\theta - 192\cot\theta\csc\theta) - (24\sec\theta - 270 + 192\csc\theta)(24\tan\theta\sec\theta)}{24^2\sec^2\theta}$

$\therefore$ At the moment: $\frac{dx}{dt} = (-0.1)(24)(-4.62978)$

$= 11.11$

Rate of change is $11.11\ cm\,s^{-1}$

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(c)(ii) $\tan\theta = \frac{192}{CK}$

$CK = \frac{192}{\tan\theta}$

$y = \frac{192}{\tan\theta} + 24$

$\frac{dy}{dt} = \left(-192\csc^2\theta\right)\frac{d\theta}{dt}$

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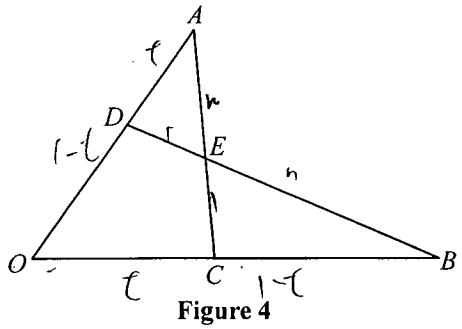

Figure 4

In Figure 4, C and D are points on OB and OA respectively such that $AD:DO = OC:CB = t:1-t$, where $0<t<1$. BD and AC intersect at E such that $AE:EC = m:1$ and $BE:ED = n:1$, where m and n are positive. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) By considering ΔOAC, express $\overrightarrow{OE}$ in terms of m, t, $\mathbf{a}$ and $\mathbf{b}$.

(ii) By considering ΔOBD, express $\overrightarrow{OE}$ in terms of n, t, $\mathbf{a}$ and $\mathbf{b}$.

(iii) Show that $m = \dfrac{t}{(1-t)^2}$ and $n = \dfrac{1-t}{t^2}$.

(iv) Chris claims that

"if $m = n$, then E is the centroid of ΔOAB".

Do you agree? Explain your answer.

(9 marks)

(b) It is given that $OA = 1$ and $OB = 2$. Francis claims that

"if AC is perpendicular to OB, then BD is always perpendicular to OA".

Do you agree? Explain your answer.

(4 marks)

(a)(i) $\overrightarrow{OC} = \dfrac{t}{t+1-t}\vec{b} = = t\vec{b}$

$\overrightarrow{AC} = t\vec{b} - \vec{a}$

$\overrightarrow{AE} = \dfrac{m}{m+1}\overrightarrow{AC} = \dfrac{tm}{m+1}\vec{b} - \dfrac{m}{m+1}\vec{a}$

$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$

$= \vec{a} + \dfrac{tm}{m+1}\vec{b} - \dfrac{m}{m+1}\vec{a}$

$= \dfrac{1}{m+1}\vec{a} - \dfrac{tm}{m+1}\vec{b}$

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(ii) $\overrightarrow{OD} = \frac{1-t}{1-t+t}\vec{a} = (1-t)\vec{a}$

$\overrightarrow{OE} = \frac{n\overrightarrow{OD} + \overrightarrow{OB}}{n+1}$

$= \frac{(n-nt)\vec{a} + \vec{b}}{n+1}$

$= \frac{n-nt}{n+1}\vec{a} + \frac{1}{n+1}\vec{b}$

(iii) $\cancel{\frac{n-nt}{n+1} = \frac{1}{m+1}}$ $\cancel{\frac{1}{m+1} = \frac{n-nt}{n+1}}$

$\cancel{m+1 = \frac{n+1}{n-nt}}$ $\cancel{\frac{1}{m+1} =}$

$\cancel{m = \frac{n+1}{n-nt} - \frac{n-nt}{n-nt}}$ $\cancel{n+1 =}$

$\cancel{m =}$

$\frac{1}{m+1} = \frac{n-nt}{n+1}$

$n+1 = (m+1)(n-nt)$

$n+1 = mn - mnt + n - nt$

$n - mn + mnt - n + nt = -1$

$n(1 - m + mt - 1 + t) = -1$

$n = \frac{-1}{-m+mt+t}$

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12. Let $M = \begin{pmatrix} k-1 & k \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$, where k and p are real numbers and $p \neq -1$.

(a) (i) Find A^{-1} in terms of p.

(ii) Show that $A^{-1}MA = \begin{pmatrix} -1 & k-p \\ 0 & k \end{pmatrix}$.

(iii) Suppose $p = k$. Using (ii), find M^n in terms of k and n, where n is a positive integer.

(8 marks)

(b) A sequence is defined by

$x_1 = 0$, $x_2 = 1$ and $x_n = x_{n-1} + 2x_{n-2}$ for $n = 3, 4, 5, \ldots$.

It is known that this sequence can be expressed in the matrix form $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$.

Using the result of (a)(iii), express x_n in terms of n.

(3 marks)

(a)(i) $\det(A) = 1+p$ $\quad A^{-1} = \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}$

(ii) $A^{-1}MA = \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k-1 & k \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{1+p}\begin{pmatrix} k-1-p & k \\ k & k \end{pmatrix}\begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{1+p}\begin{pmatrix} -1-p & pk-p-p^2+k \\ 0 & pk+k \end{pmatrix}$

$= \begin{pmatrix} -1 & k-p \\ 0 & k \end{pmatrix}$

(iii) when $p=k$, $A^{-1}MA = \begin{pmatrix} -1 & 0 \\ 0 & k \end{pmatrix}$

$(A^{-1}MA)^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & k^n \end{pmatrix}$

$A(A^{-1}MA)^n A^{-1} = \begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & k^n \end{pmatrix} \cdot \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}$

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(a)(iii) $\therefore$ $M^n = \begin{pmatrix} 1 & k \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & k^n \end{pmatrix} \frac{1}{1+k} \begin{pmatrix} 1 & -k \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} (-1)^n & k^{n+1} \\ (-1)^{n+1} & k^n \end{pmatrix} \frac{1}{1+k} \begin{pmatrix} 1 & -k \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{1+k} \begin{pmatrix} (-1)^n + k^{n+1} & k(-1)^{n+1} + k^{n+1} \\ (-1)^{n+1} + k^n & (-1)^{n+2} k + k^n \end{pmatrix}$

(b) $\cancel{\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} x_{n-1} + 2x_{n-2} \\ x_{n-1} \end{pmatrix}}$

sub $k=2$, $M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = M \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$

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13. (a) Prove that $1-\cos 4\theta - 2\cos 2\theta \sin^2 2\theta = 16\cos^2\theta \sin^4\theta$.

(2 marks)

(b) Show that $\int_0^{n\pi} \cos^2 x \sin^4 x \, dx = \frac{n\pi}{16}$, where n is a positive integer.

(4 marks)

(c) Let $f(x)$ be a continuous function such that $f(k-x) = f(x)$, where k is a constant.

Show that $\int_0^k x f(x)\, dx = \frac{k}{2}\int_0^k f(x)\, dx$.

(4 marks)

(d)

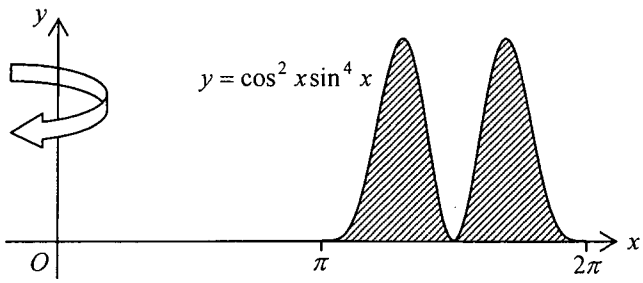

Figure 5

Figure 5 shows the shaded region bounded by the curve $y = \cos^2 x \sin^4 x$ and the x-axis, where $\pi \le x \le 2\pi$. Find the volume of the solid of revolution when the shaded region is revolved about the y-axis.

(4 marks)

(a) $1-\cos 4\theta - 2\cos 2\theta \sin^2 2\theta$

$= 1 - 1 + 2\sin^2 2\theta - 2\cos 2\theta \sin^2 2\theta$

$= 2\sin^2 2\theta (1 - \cos 2\theta)$

$= 2(4\sin^2\theta\cos^2\theta)(1 - 1 + 2\sin^2\theta)$

$= 8\sin^2\theta\cos^2\theta \, 2\sin^2\theta$

$= 16\cos^2\theta\sin^4\theta$

(b) $\int_0^{n\pi} \cos^2 x \sin^4 x \, dx = \frac{1}{16}\int_0^{n\pi} 16\cos^2 x \sin^4 x \, dx$

$= \frac{1}{16}\int_0^{n\pi} (1 - \cos 4x - 2\cos 2x \sin^2 2x)\, dx$

$= \frac{1}{16}\int_0^{n\pi} 1\, dx - \frac{1}{64}\int_0^{n\pi} \cos 4x \, d(4x) - \frac{1}{16}\int_0^{n\pi} \cos 2x \sin^2 2x \, d(2x)$

$= \frac{1}{16}[x]_0^{n\pi} - \frac{1}{64}[\sin 4x]_0^{n\pi} - \frac{1}{16}\int_0^{n\pi} \sin^2 2x \, d(\sin 2x)$

$= \frac{n\pi}{16} - \frac{1}{64}(0) - \frac{1}{48}[\sin^3 2x]_0^{n\pi}$

$= \frac{n\pi}{16}$

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$\int x^2 dx$

$= \frac{x^3}{3}$

(c) let $u = k-x$ when $x=0$, $u=k$ when $x=k$, $u=0$

$$\int_0^k xf(x)\,dx = -\int_k^0 (k-u)f(k-u)\,du$$

$$= \int_0^k (k-u)f(u)\,du$$

$$= \int_0^k (k-x)f(x)\,dx$$

$$= \int_0^k kf(x)\,dx - \int_0^k xf(x)\,dx$$

$$\therefore \int_0^k xf(x)\,dx = k\int_0^k f(x)\,dx - \int_0^k xf(x)\,dx$$

$$2\int_0^k xf(x)\,dx = k\int_0^k f(x)\,dx$$

$$\int_0^k xf(x)\,dx = \frac{k}{2}\int_0^k f(x)\,dx$$

(d) $$\text{Volume} = 2\pi\int_\pi^{2\pi} x\cos^2 x\sin^4 x\,dx$$

let $f(x) = \cos^2 x\sin^4 x$

$$f(\pi-x) = (-\cos x)^2\sin^4 x$$
$$= \cos^2 x\sin^4 x$$
$$= f(x)$$

$$f(2\pi-x) = \cos^2 x(-\sin x)^4$$
$$= \cos^2 x\sin^4 x$$
$$= f(x)$$

$$\therefore \int_0^\pi x\cos^2 x\sin^4 x\,dx = \frac{\pi}{2}\int_0^\pi \cos^2 x\sin^4 x\,dx$$
$$= \frac{\pi}{2}\left(\frac{\pi}{16}\right)$$
$$= \frac{\pi^2}{32}$$

$$\int_0^{2\pi} x\cos^2 x\sin^4 x\,dx = \frac{2\pi}{2}\int_0^{2\pi} \cos^2 x\sin^4 x\,dx$$
$$= \pi\left(\frac{2\pi}{16}\right)$$
$$= \frac{\pi^2}{8}$$

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$\therefore$ Volume $= 2\pi \int_{\pi}^{2\pi} x \cos^2 x \sin^4 x \, dx$

$= 2\pi \left[\int_0^{2\pi} x \cos^2 x \sin^4 x \, dx - \int_0^{\pi} x \cos^2 x \sin^4 x \, dx \right]$

$= 2\pi \left(\frac{\pi^2}{8} - \frac{\pi^2}{32} \right)$

$= 2\pi \left(\frac{3\pi^2}{32} \right)$

$= \frac{3\pi^3}{16}$ //

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END OF PAPER

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Comments

The candidate demonstrates comprehensive knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by applying them successfully at a sophisticated level to a wide range of unfamiliar situations in Questions 10, 12 and 13.

He/She is able to communicate and express views and arguments precisely and logically using mathematical language, notations and diagrams, such as binomial coefficients in Question 1, limit notations and a derivative test table in Question 2, vector notations in Question 8, and matrix notations in Questions 9 and 12.

The candidate also provides complex mathematical proofs in a logical, rigorous and concise manner in proving identities as in Questions 7(a), 12(b) and 13.

He/She is able to integrate knowledge and skills from different areas of the curriculum in handling complex tasks using a variety of strategies, such as differentiation and coordinate geometry in Question 3, vectors and three dimensional geometry in Question 8, and trigonometry and integration in Question 13.